On the Surjectivity of Galois Representations Associated to Elliptic Curves over Number Fields

Eric Larson and Dmitry Vaintrob

Abstract

Given an elliptic curve E over a number field K, the ℓ-torsion points $E[\ell]$ of E define a Galois representation $\mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_2(\mathbb{F}_\ell)$. A famous theorem of Serre [9] states that as long as E has no Complex Multiplication (CM), the map $\mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_2(\mathbb{F}_\ell)$ is surjective for all but finitely many ℓ.

We say that a prime number ℓ is *exceptional* (relative to the pair (E, K)) if this map is *not* surjective. Here we give a new bound on the largest exceptional prime, as well as on the product of all exceptional primes of E. We show in particular that conditionally on the Generalized Riemann Hypothesis (GRH), the largest exceptional prime of an elliptic curve E without CM is no larger than a constant (depending on K) times $\log N_E$, where N_E is the absolute value of the norm of the conductor. This answers affirmatively a question of Serre in [10].

1 Introduction

Let E be an elliptic curve over a number field K, and for each prime number ℓ, let $E[\ell]$ be the group of ℓ-torsion points of E over $\overline{K}$. This group is isomorphic to $(\mathbb{Z}/\ell\mathbb{Z})^2$ and has action by the absolute Galois group $G_K := \mathrm{Gal}(\overline{K}/K)$, which we denote

$$\rho_{E,\ell}\colon G_K \to \mathrm{GL}(E[\ell]) \simeq \mathrm{GL}_2(\mathbb{F}_\ell).$$

The collection of representations $\rho_{E,\ell}$ encode many important properties of E, such as its primes of bad reduction and its number of points over finite fields.

As long as E has no complex multiplication (CM), these representations are surjective for all but finitely many ℓ, which we call *exceptional primes* for E. This result was proven in Serre's 1968 paper [9], and concluded the proof of the long-conjectured Open Image Theorem — the statement that the inverse limit of the images

$$\varprojlim_{m \in \mathbb{Z}} \rho_{E,m}(G_K) \subset \varprojlim_m \mathrm{GL}_2(\mathbb{Z}/m\mathbb{Z})$$

has finite index in $\varprojlim_m \mathrm{GL}_2(\mathbb{Z}/m\mathbb{Z}) \cong \mathrm{GL}_2(\widehat{\mathbb{Z}})$.

Serre's original proof was ineffective, even over the ground field $\mathbb{Q}$. But in the later paper [10], he gave in the case of $K = \mathbb{Q}$ an explicit upper bound on the largest exceptional prime of an elliptic curve E over the rational numbers without CM, conditionally on the Generalized Riemann Hypothesis (GRH). Namely he showed that the largest exceptional prime ℓ_E is bounded by the following expression in the conductor N_E of the elliptic curve:

$$\ell_E \leq C_1 \cdot \log N_E \cdot (\log\log N_E)^3, \tag{1}$$

for C_1 an absolute (and effectively computable) constant. In the same paper, he conjectured that, conditionally on GRH, a similar bound should hold for elliptic curves defined over arbitrary fields K.

An effective bound over arbitrary number fields K was later given, unconditionally, by the paper of Masser and Wüstholz [8], with bound $C_2 \cdot \max(h_E, n_K)^\gamma$ for absolute constants C_2 and γ, where h_E is the logarithmic height of the j-invariant of E and n_K is the degree of K. Here, the constant γ is very large (although it can be reduced to 2 if we only care about bounding degrees of isogenies). Our results imply that conditionally on GRH, we can take $\gamma = 1$ if we let C_2 depend on K.

Over $\mathbb{Q}$, Kraus and Cojocaru ([4] and [2]) gave another unconditional bound in terms of the conductor using the modularity of elliptic curves over $\mathbb{Q}$, namely

$$\ell_E \leq C_3 \cdot N_E \cdot (\log\log N_E)^{1/2}.$$

Moreover, in [12], Zywina shows that the product

$$A_E := \prod_{\ell \text{ exceptional for } E} \ell$$

can be bounded by the b_Eth power of each of the above bounds on ℓ_E, where b_E is the number of primes of bad reduction for E.

The gradual improvements in the bound on exceptional primes have paid off. A recent paper of Bilu and Parent [1] which made a breakthrough in the search for a uniform bound on exceptional primes over $\mathbb{Q}$ (showing that $X_{\text{split}}(\ell)(\mathbb{Q})$ consists only of CM points and cusps for ℓ sufficiently large) relied crucially the value of γ appearing in the Masser-Wüstholz bound.

This paper continues this tradition. We bound, conditionally on GRH, both the largest exceptional prime ℓ_E and the product of all exceptional primes A_E. Our proof is in the spirit of Serre's original bound in [10], but we allow E to be defined over an arbitrary number field K, which entails a more delicate analysis. The bound on the largest exceptional prime we get is, as conjectured in [10], the same as what Serre obtained when $K = \mathbb{Q}$ (equation (1), with the constant C_1 replaced by a constant $C(K)$ depending on the number field K).

In fact, for fixed ground field K, we show that an asymptotically better bound holds. Namely, conditionally on GRH, the largest exceptional prime ℓ_E satisfies

$$\ell_E \leq C'(K) \cdot \log N_E,$$

where N_E is the absolute value of the norm of the conductor of E.

We make the constant $C(K)$ in our first bound effective, but have at the moment no effective way of determining the constant $C'(K)$ in the second, asymptotically better bound (even over $K = \mathbb{Q}$).

We also give a conditional bound on the product of all exceptional primes, A_E. We show, in particular, that for fixed K and fixed $\epsilon > 0$, we have $A_E < N_E^{\epsilon}$ for all but finitely many curves E. The bound one would get by multiplying together all primes up to our upper bound for ℓ_E — as well as bounds on A_E given in earlier papers — give values which are asymptotic to a positive power of N_E.

For the remainder of the paper, we assume the Generalized Riemann Hypothesis (GRH).

Our proof can be roughly outlined as follows. First we compare an exceptional prime ℓ and an *unexceptional* prime p, and show that the two Galois representations $\rho_{E,\ell}$ and $\rho_{E,p}$ impose conditions on traces of Frobenius of E which are incompatible if ℓ is sufficiently large compared to p and N_E. This part relies on the effective Chebotarev Theorem of Lagarias and Odlyzko together with a result of our earlier paper [6].

Next, we give an upper bound for the *smallest unexceptional* prime p. The analysis here bifurcates. Ineffectively, it can be easily shown that the smallest such p is bounded above by a constant depending only on K. The effective bound is trickier, and uses Serre's method in [10], which depends on GRH in an essential way.

Combining the bound on the unexceptional p with the bound on the exceptional ℓ in terms of p completes the proof. We then show that the bound on ℓ can be tweaked to give an upper bound on the product A_E of all exceptional primes. Throughout the paper, we treat separately two different kinds of exceptional primes: those for which $\rho_{E,\ell}$ is absolutely irreducible, and those for which it is not. While the analysis in the two cases is remarkably parallel, our bound on the product of exceptional primes ℓ of the second kind (such that $\rho_{E,\ell}$ is reducible over $\overline{\mathbb{F}}_\ell$) turns out to be significantly better, polynomial in $\log N_E$ (see Lemma 17).

Fix a number field K, and write n_K, r_K, R_K, h_K, and Δ_K for the degree, rank of the unit group, regulator, class number, and discriminant of K respectively. Let us choose for every prime ideal v of K, a corresponding Frobenius element $\pi_v \in G_K := \mathrm{Gal}(\overline{K}/K)$. We let E be an elliptic curve *without complex multiplication (CM)*, and we write N_E and a_E for the absolute value of the norm of the conductor of E, and the number of primes of additive reduction of E, respectively. We say that $X \ll_K Y$ if there are *effectively computable* constants A and B depending only on K for which $X \leq AY + B$. Moreover, we say that $X \lll_K Y$ if $X \leq AY + B$ for constants A and B that are *not* assumed to be effectively computable. If the constants A and B are absolute, we drop the K subscript on the $\ll$ and $\lll$. With this notation, our results are as follows.

Theorem 1 (Theorem 23)**.** *Let E be an elliptic curve over a number field K without CM. Then any exceptional prime ℓ satisfies*

$$\ell \lll_K \log N_E.$$

Moreover, the product of all exceptional primes satisfies

$$\prod \ell \lll_K 4^{a_E} \cdot (\log N_E)^{14}.$$

Theorem 2 (Theorem 25)**.** *Let E be an elliptic curve over a number field K without CM. Then any exceptional prime ℓ satisfies*

$$\ell \ll_K \log N_E \cdot (\log\log N_E)^3.$$

Moreover, the product of all exceptional primes satisfies

$$\prod \ell \ll_K 4^{a_E} \cdot (\log N_E)^{14} \cdot (a_E + \log\log N_E)^6 \cdot (\log\log N_E)^{36} \ll_K 4^{a_E} \cdot (\log N_E)^{21},$$

where a_E is the number of primes of K of additive reduction for E.

Acknowledgements

We would like to thank David Zureick-Brown, Bryden Cais and Ken Ono for suggesting us the problem that led to this paper and answering questions. Thanks also to Barry Mazur for valuable comments and discussions. We would also like to express our admiration for Serre's papers [9] and [10], whose techniques provide the mathematical inspiration and departure-point of our work.

2 Possible Images of the Representation $\rho_{E,\ell}$

In this section, we analyze the possible images of $\rho_{E,\ell}$. The proofs of all of the results of this section are in the papers [9] and [10] by Serre. We begin by singling out some subgroups of $\mathrm{GL}_2(\mathbb{F}_\ell)$.

Definition 3. A *Cartan* subgroup is a subgroup of $\mathrm{GL}_2(\mathbb{F}_\ell) \subset \mathrm{GL}_2(\overline{\mathbb{F}}_\ell)$ which fixes two one-dimensional subspaces of $\overline{\mathbb{F}}_\ell^2$, i.e. which in some basis of $\overline{\mathbb{F}}_\ell^2$ looks like

$$\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}.$$

A Cartan subgroup is index two in its normalizer. The normalizer consists of matrices of the form

$$\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}$$

(i.e. matrices which either fix or permute the two subspaces fixed by the Cartan subgroup).

Lemma 4. *Let G be any subgroup of $\mathrm{GL}_2(\mathbb{F}_\ell)$. Then, one of the following holds:*

1. *(Reducible Case) G acts reducibly on $\overline{\mathbb{F}}_\ell^2$.*

2. *(Normalizer Case) G is contained in the normalizer of a Cartan subgroup, but not in the Cartan subgroup itself.*

3. *(Special Linear Case) G contains $\mathrm{SL}_2(\mathbb{F}_\ell)$.*

4. *(Irregular Case) The image of G under the projection $\mathrm{GL}_2(\mathbb{F}_\ell) \to \mathrm{PGL}_2(\mathbb{F}_\ell)$, is contained in a subgroup which is isomorphic to A_4, S_4, or A_5.*

Remark 5. We use the term "irregular" subgroup to avoid a clash of notation; usually they are called "exceptional" subgroups.

Proof. See Section 2 of [9]. ☐

Definition 6. Having fixed the field K, we call a prime number p *acceptable* if p is unramified in $K/\mathbb{Q}$ and $p \geq 53$. (So almost all primes are acceptable.) For the remainder of the paper, we will only consider acceptable primes.

Lemma 7. *If p is acceptable, then $\mathbb{P}\rho_{E,p}$ contains an element of order at least* 13.

Proof. This follows from Lemma 18′ of [10] (which is stated for $K = \mathbb{Q}$, but the same proof works as long as p is unramified in K). ☐

Lemma 8. *Let ℓ be an acceptable exceptional prime. Then the image of $\rho_{E,\ell}$ falls into either the reducible case or the normalizer case of Lemma 4.*

Proof. Since $\ell \nmid \Delta_K$, it follows that $\det \rho_{E,\ell}$ is surjective, so the image of $\rho_{E,\ell}$ cannot fall into case 3 because ℓ is exceptional. By Lemma 7, the image of $\rho_{E,\ell}$ cannot fall into case 4. ☐

The two remaining cases will require separate analysis, and throughout the paper we will separate them as the "reducible" case and the "normalizer" case.

3 The Effective Chebotarev Theorem

We have the following effective version of the Chebotarev Density Theorem, due to Lagarias and Odlyzko.

Theorem 9 (Effective Chebotarev)**.** *Let L/K be a Galois extension of number fields with $L \neq \mathbb{Q}$. Then every conjugacy class of $\mathrm{Gal}(L/K)$ is represented by the Frobenius element of a prime ideal v such that*

$$\mathrm{Nm}_{\mathbb{Q}}^{K}(v) \ll (\log \Delta_E)^2.$$

Proof. See [5], remark at end of paper regarding the improvement to Corollary 1.2. ∎

Corollary 10 (Effective Chebotarev with avoidance). *Let L/K be a Galois extension of number fields with $L \neq \mathbb{Q}$ and $\Sigma \subset \Sigma_K$ a finite set of primes which includes the primes at which L/K is ramified. Let N be the norm of the product of the primes of Σ, and write $d = [L : K]$. Then every conjugacy class of $\operatorname{Gal}(L/K)$ is represented by the Frobenius element of a prime ideal $v \in \Sigma_K \smallsetminus \Sigma$ such that*

$$\operatorname{Nm}_{\mathbb{Q}}^{K}(v) \ll d^2 \cdot \big(\log N + \log \Delta_K + n_K \log d\big)^2 \ll_K d^2 \cdot \big(\log N + \log d\big)^2.$$

Proof. Let H be the Hilbert class field of K, of degree h_K over K. Then $\Delta_H = \Delta_K^{h_K}$, so any element of the class group is represented by a prime ideal $v \in \Sigma_K$ of norm $\ll (h_K \log \Delta_K)^2$. It follows from a result of Lenstra (Theorem 6.5 in [7]) that $h_K \leq \Delta_K^{3/2}$, so we can take $\operatorname{Nm}(v) \ll \Delta_K^4$. Now, we let $I = \prod_{v \in \Sigma} v$, and apply this result to the image in the class group of the ideal I^{-1}. We get a prime ideal v_0 with $\operatorname{Nm}(v_0) \ll \Delta_K^4$ such that $v_0 I$ is principal, generated by $x \in K$.

Define $L' = L[\sqrt[3]{x}, \omega]$, for a primitive cube root of unity ω. The set $\Sigma' \subset \Sigma_K$ of primes ramified in L'/K consists of all elements of Σ, plus some primes dividing $6v_0$.

Now, we apply effective Chebotarev again, to $\operatorname{Gal}(L'/K)$, to conclude that every conjugacy class of $\operatorname{Gal}(L'/K)$ is represented by a Frobenius element of a prime ideal $v \in \Sigma_K$ which is unramified in L', and thus not in Σ, with

$$\operatorname{Nm}_{\mathbb{Q}}^{K}(v) \ll (\log \Delta_{L'})^2.$$

We now turn to bounding $\log \Delta_{L'}$. For a prime v of K, write e_v and f_v for the ramification and inertial degrees of v respectively. We have

$$\begin{aligned}
\log \Delta_{L'} &= [L' : K] \cdot \log \Delta_K + \log \operatorname{Nm}_K^{L'} \mathfrak{d}_K^{L'} \\
&\leq 6d \log \Delta_K + \sum_{v \in \Sigma'} \left((6d-1) f_v \log p_v + 6d f_v e_v \operatorname{val}_{p_v}(d) \log p_v\right) \\
&\leq 6d \cdot \left(\log \Delta_K + \sum_{v \in \Sigma'} f_v \log p_v + \sum_{v \in \Sigma'} f_v e_v \operatorname{val}_{p_v}(d) \log p_v\right) \\
&\leq 6d \cdot \big(\log \Delta_K + \log(N \cdot 6\Delta_K^4) + n_K \log d\big) \\
&\ll d \cdot (\log N + \log \Delta_K + n_K \log d).
\end{aligned}$$

∎

Throughout the paper, we will frequently apply the above corollary to Galois representations built out of the representations $\rho_{E,\ell}$. For this purpose, recall the well-known Néron-Ogg-Shafarevich criterion:

Theorem 11 (Néron-Ogg-Shafarevich). *Let E be an elliptic curve over K. Then $\rho_{E,\ell}$ is ramified only at primes dividing ℓ and the conductor of E.*

Proof. This is well known; see e.g. Proposition 4.1 of [11]. ∎

4 Bounds In Terms of The Smallest Unexceptional Prime

Recall that we have fixed an elliptic curve E over a number field K, and ℓ is an exceptional prime for (E, K). In this section we give bounds on both the largest exceptional prime and the product of all exceptional primes, in terms of the smallest *unexceptional* prime.

4.1 The Reducible Case

Suppose that $E[\ell]$ is reducible over $\overline{\mathbb{F}}_\ell$, and write

$$\rho_{E,\ell} \otimes_{\mathbb{F}_\ell} \overline{\mathbb{F}}_\ell = \begin{pmatrix} \psi_\ell^{(1)} & - \\ 0 & \psi_\ell^{(2)} \end{pmatrix}.$$

Theorem 12. *There exists a finite set S_K of primes numbers depending only on K such that if $\ell \notin S_K$, then there exists a CM elliptic curve E', which is defined over K and whose CM-field is contained in K, such that for some character ϵ_ℓ:* $\mathrm{Gal}(\overline{K}/K) \to \mu_{12}$,

$$\begin{cases} \psi_\ell^{(1)} & = \varphi_\ell^{(1)} \otimes \epsilon_\ell \\ \psi_\ell^{(2)} & = \varphi_\ell^{(2)} \otimes \epsilon_\ell^{-1} \end{cases} \quad \textit{where} \quad \rho_{E',\ell} \otimes_{\mathbb{F}_\ell} \overline{\mathbb{F}}_\ell = \begin{pmatrix} \varphi_\ell^{(1)} & 0 \\ 0 & \varphi_\ell^{(2)} \end{pmatrix}. \tag{2}$$

Moreover the elliptic curve E' depends only on E (i.e. is independent of ℓ), and ϵ_ℓ is ramified only at primes dividing the conductor of E.

Proof. See Theorem 1 of [6] and Remark 1.1 following the theorem. ☐

This lets us relate the Frobenius polynomials of E and E' at small primes of K. We make the following definitions.

Definition 13. Fix E and E' as above. We define R_E to be the product of all reducible primes ℓ satisfying equation (2).

The fact that E' depends only on E (for $\ell \gg_K 1$) implies that

$$\prod_{\rho_{E,\ell} \text{ reducible}} \ell \ll_K R_E.$$

(Moreover, this is sharp, as R_E divides the product on the left.)

Definition 14. For a polynomial $P \in \mathbb{Z}[x]$, define its $12th$ *Adams operation* $\Psi^{12}P \in \mathbb{Z}[x]$ to be the polynomial whose roots (in $\mathbb{C}$) are the twelfth powers of the roots of P.

Using this notation and writing

$$P_E(v) = x^2 + \mathrm{Tr}_E(\pi_v)x + \mathrm{Nm}(v)$$

for the Frobenius polynomial of $\pi_v \in G_K$, we have the following result (where E' is the CM elliptic curve from above).

Lemma 15. *Let v be a prime of K at which E has good reduction. If $4(\mathrm{Nm}\, v)^6 < R_E$, then*

$$\Psi^{12} P_E(v) = \Psi^{12} P_{E'}(v);$$

moreover, if $\ell \mid R_E$ is such that $4\sqrt{\mathrm{Nm}\, v} < \ell$ and $\epsilon_\ell(\pi_v) = 1$ (where $\epsilon_\ell \colon G_K \to \mu_{12}$ is as in Theorem 12), then

$$P_E(v) = P_{E'}(v).$$

Proof. Suppose $\ell \mid R_E$, i.e. ℓ satisfies equation (2). In particular, $(\psi_\ell^{(1)})^{12} = (\varphi_\ell^{(1)})^{12}$ and $(\psi_\ell^{(2)})^{12} = (\varphi_\ell^{(2)})^{12}$, i.e. $\Psi^{12} P_E \equiv \Psi^{12} P_{E'} \bmod \ell$. Since this holds for all $\ell \mid R_E$, by plugging in v we obtain

$$\Psi^{12} P_E(v) \equiv \Psi^{12} P_{E'}(v) \mod R_E.$$

If moreover $\epsilon_\ell(\pi_v) = 1$, then $\psi_\ell^{(1)}(\pi_v) = \varphi_\ell^{(1)}(\pi_v)$ and $\psi_\ell^{(2)}(\pi_v) = \varphi_\ell^{(2)}(\pi_v)$. Equivalently,

$$P_E(v) \equiv P_{E'}(v) \mod \ell.$$

From the Weil bounds, $P_{E_0}(v)$ has nonpositive discriminant and constant term $\mathrm{Nm}\, v$ for any elliptic curve E_0 and prime v of good reduction for E_0. In other words,

$$P_{E_0}(v) = x^2 - ax + \mathrm{Nm}\, v \quad \text{and} \quad \Psi^{12} P_{E_0}(v) = x^2 - bx + \mathrm{Nm}\, v^{12},$$

with $|a| \le 2\sqrt{\mathrm{Nm}\, v}$ and $|b| \le 2(\mathrm{Nm}\, v)^6$. It follows that $P_E(v) - P_{E'}(v) = Ax$ for some $|A| \le 4\sqrt{\mathrm{Nm}\, v}$ and $\Psi^{12} P_E - \Psi^{12} P_{E'} = Bx$ for some $|B| \le 4(\mathrm{Nm}\, v)^6$. On the other hand, we have seen above that $\ell \mid A$ and $R_E \mid B$. The lemma follows, using that $|A| < \ell$ and $\ell \mid A$ imply $A = 0$ (and similarly for B). $\square$

Now we are in a position to bound any prime ℓ with reducible $\rho_{E,\ell}$ (or the product of all such) in terms of a small unexceptional prime p.

Lemma 16. *Suppose that p is an acceptable prime that does not divide R_E. Let E' be as above, and let $H \subset \mathrm{GL}_2(\mathbb{F}_p) \times \mathrm{GL}_2(\mathbb{F}_p)$ be the image of $\rho_{E,p} \times \rho_{E',p}$. Then there exists a surjection $f \colon H \twoheadrightarrow G$ with $|G| \ll p^3$, and a $g \in G$ such that for any $(X, Y) \in H$ with $f(X, Y) = g$, we have $\mathrm{Tr}(X^{12}) \neq \mathrm{Tr}(Y^{12})$.*

Proof. First suppose that p is unexceptional. By the theory of complex multiplication, the image of $\rho_{E',p}$ is contained in either a split or a nonsplit Cartan subgroup. Hence, the image of the projectivization $\mathbb{P}\rho_{E',p}$ is contained in a cyclic group of order $p \pm 1$. Since p is acceptable, $p \pm 1 \nmid 12$. It follows that there is an $M \in \mathrm{PGL}_2(\mathbb{F}_p)$ whose 12th power is not conjugate to anything in the image of $\mathbb{P}\rho_{E',p}$. Taking $G = \mathrm{PGL}_2(\mathbb{F}_p)$ and f to be projection onto the first factor followed by the projection $\mathrm{GL}_2(\mathbb{F}_p) \to \mathrm{PGL}_2(\mathbb{F}_p)$ completes the proof in this case.

Hence we can assume that p is of normalizer type, or of reducible type but not satisfying the condition (2) with respect to the CM curve E'. Write $\widetilde{\rho}_{E,p}$ for the semisimplification

(i.e. direct sum of the Jordan-Holder quotients) of $\rho_{E,p}$. To bound the product of all exceptional primes of E, we consider the Galois representation

$$\Pi = \widetilde{\rho}_{E,p} \times \rho_{E',p} \colon G_K \to \mathrm{GL}_2(\mathbb{F}_p) \times \mathrm{GL}_2(\mathbb{F}_p).$$

Since $\det \rho_{E,p} = \det \rho_{E',p}$ is surjective onto $\mathbb{F}_p^\times$ and either Cartan or Normalizer subgroups of $\mathrm{GL}_2(\mathbb{F}_p)$ have $\ll p^2$ elements, the image has order $\ll p^3$.

Now we claim that the image of Π contains something of the form (X, Y) for which $\operatorname{Tr} X^{12} \neq \operatorname{Tr} Y^{12}$. If p is of reducible type, then this is clear by the assumption that $p \nmid R_E$. If p is of normalizer type, then by Lemma 7, the projective image of $\mathbb{P}\rho_{E,p}$ contains an element of order at least 13. In particular, it must contain something of the form

$$A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix},$$

with $a^{12} \neq b^{12}$. Let B be an element of $\mathrm{GL}_2(\mathbb{F}_p)$ in the image of $\rho_{E,p}$ that is not in the Cartan group. Since the image of $\rho_{E',p}$ is abelian, it follows that the image of Π contains $(1, M)$, where $M = ABA^{-1}B^{-1}$. By explicit computation,

$$M = \begin{pmatrix} a^{-1}b & 0 \\ 0 & b^{-1}a \end{pmatrix}.$$

Taking $X = 1$ and $Y = M$ thus completes the proof. □

Lemma 17. *Let p be the smallest acceptable prime that does not divide R_E.*

$$R_E \ll_K p^{36} \cdot (\log N_E + \log p)^{12}.$$

Moreover, any prime $\ell \mid R_E$ satisfies

$$\ell \ll_K p^3 \cdot (\log N_E + \log p).$$

Proof. Let $f \colon H \twoheadrightarrow G$ and $g \in G$ be as in Lemma 16.

First, we bound R_E. By Corollary 10 applied to $g \in G$, Néron-Ogg-Shafarevich, and Lemma 16, there is a prime v of good reduction for E such that $\operatorname{Tr} \rho_{E,p}(\pi_v^{12}) \neq \operatorname{Tr} \rho_{E',p}(\pi_v^{12})$, which moreover satisfies

$$\mathrm{Nm}\, v \ll_K p^6 \cdot (\log N_E + \log p)^2. \tag{3}$$

In particular, $\Psi^{12} P_E(v) \neq \Psi^{12} P_{E'}(v)$, so by Lemma 15 we have

$$R_E \leq 4(\mathrm{Nm}\, v)^6 \ll_K p^{36} \cdot (\log N_E + \log p)^{12}.$$

To bound the largest exceptional prime, we consider the direct sum of ϵ_ℓ and G. Since ϵ_ℓ has order 12, the image of this Galois representation contains $(1, g^{12})$. Applying Corollary 10 to $g^{12} \in G$, Néron-Ogg-Shafarevich, and Lemma 16, we can find a prime v of good reduction for E such that $\operatorname{Tr} \rho_{E,p}(\pi_v) \neq \operatorname{Tr} \rho_{E',p}(\pi_v)$ and $\epsilon_\ell(\pi_v) = 1$, which satisfies the bound (3). In particular, $P_E(v) \neq P_{E'}(v)$, so Lemma 15 gives $\ell \leq 4\sqrt{\mathrm{Nm}\, v} \ll_K p^6 \cdot (\log N_E + \log p)$, as desired. □

4.2 The Normalizer Case

Let ℓ be a prime such that the image of $\rho_{E,\ell}$ falls into the normalizer case of Lemma 4. Write C for our Cartan subgroup and N for its normalizer. Then we have a quadratic character χ on $\mathrm{Gal}(\overline{\mathbb{Q}}/K)$ given by

$$\chi\colon\ \mathrm{Gal}(\overline{\mathbb{Q}}/K) \to N \to N/C \simeq \{\pm 1\}.$$

Lemma 18. *The character χ is ramified only at places of bad additive reduction.*

Proof. See Lemma 2 in Section 4.2 of [9]. ☐

In this case, we say that ℓ is *χ-exceptional (of normalizer type).* More generally, if $V \subset \hom(G_K, \mathbb{Z}/2)$ is an $\mathbb{F}_2$-vector space of Galois characters, we say that ℓ is V-exceptional if ℓ is χ-exceptional for some $\chi \in V$. Note that the space V of characters induces a Galois extension of K with Galois group the dual $\mathbb{F}_2$-vector space V^*, via the following construction.

Definition 19. For $V \subset \hom(G_K, \mathbb{Z}/2\mathbb{Z})$, we write $\rho_V\colon G_K \to V^*$ for the map induced by the pairing $V \times G_K^{\mathrm{ab}} \to \mathbb{F}_2$.

This gives (functorially) a one-to-one correspondence between finite $\mathbb{F}_2$-vector spaces of Galois characters and finite abelian field extensions with Galois group annihilated by 2.

Lemma 20. *The vector space V of all quadratic Galois characters ramified only at places of bad additive reduction satisfies*

$$|V| \leq 2^{a_E+2n_K} \cdot h_K.$$

(In fact, the argument below shows $|V| \leq 2^{a_E+2n_K} \cdot 2^{r_2(\mathrm{Cl}(K))}$, where $r_2(\mathrm{Cl}(K))$ is the 2-rank of the class group.)

Proof. Note that $|V| = |V^*|$. Write U_K for the subgroup of principal idèles in the group $\mathbb{I}_K$ of idèles. By class field theory, ρ_V induces a surjection $\mathbb{I}_K \to V^*$. Since $[\mathbb{I}_K : U_K] = h_K$, it suffices to show that $\rho_V(U_K) \subset V$ has order at most $2^{a_E+n_K}$. However, the restriction $\rho_V|_{U_K}$ factors through the projection

$$U_K \to \prod_{v \text{ of additive reduction}} \mathcal{O}_v^*/(\mathcal{O}_v^*)^2.$$

Now by a standard application of Hensel's lemma, if $p_v \neq 2$ then $\mathcal{O}_v^*/(\mathcal{O}_v^*)^2 = \mathbb{F}_2$, and if $p_v = 2$ then $\mathcal{O}_v^*/(\mathcal{O}_v^*)^2$ is a vector space over $\mathbb{F}_2$ of dimension at most $2e_vf_v$. Since $\sum_{v|2} 2e_vf_v = 2n_K$, this gives the desired bound. ☐

Lemma 21. *Let V be a d-dimensional vector space of quadratic Galois characters ramified only at places of bad additive reduction, and let p be the smallest acceptable prime that is not V-exceptional. Then the product of all V-exceptional primes ℓ satisfies*

$$\prod \ell \ll_K \left(2^d \cdot p^3 \cdot (\log N_E + \log p)\right)^{2-2^{1-d}}.$$

Proof. We start by showing that for any $\alpha \in V^*$, there is some $X_\alpha \in \mathrm{PGL}_2(\mathbb{F}_p)$ of nonzero trace such that (α, X_α) is contained in the image of $\rho_V \times \mathbb{P}\rho_{E,p}$.

If p is unexceptional, then $\mathbb{P}\rho_{E,p}$ surjects onto $\mathrm{PGL}_2(\mathbb{F}_p)$. Hence, the abelianization of $\mathbb{P}\rho_{E,p}$ is the quadratic character defined by $\mathrm{PGL}_2(\mathbb{F}_p)/\mathrm{PSL}_2(\mathbb{F}_p)$. Since V^* is an abelian group, the image of $\rho_V \times \mathbb{P}\rho_{E,p}$ contains everything of the form (α, X) either for every $X \in \mathrm{PSL}_2(\mathbb{F}_p)$, or for every $X \notin \mathrm{PSL}_2(\mathbb{F}_p)$. Either way, the image contains something of the form (α, X_α) where X_α has nonzero trace.

If p is exceptional, then since p is acceptable, p is either of normalizer or of reducible type. Pick some Y_α so that $(\alpha, Y_\alpha) = (\rho_V \times \rho_{E,\ell})(g_\alpha)$ is in the image of $\rho_V \times \mathbb{P}\rho_{E,p}$. If p is exceptional of normalizer type, then since p is not V-exceptional, we can choose Y_α so that Y_α lies in the Cartan subgroup. If $\mathrm{Tr}(Y_\alpha) \neq 0$, we are done, so suppose $\mathrm{Tr}(Y_\alpha) = 0$. From Lemma 7, there is an element $Z_\alpha = \mathbb{P}\rho_{E,p}(h_\alpha)$ of order greater than four in the image of $\rho_{E,p}$ (which must lie in the Cartan subgroup). Now we can take $X_\alpha = Y_\alpha Z_\alpha^2$ which has nonzero trace and satisfies $(\rho_V \times \rho_{E,p})(g_\alpha h_\alpha^2) = (\alpha, Y_\alpha Z_\alpha^2)$. as desired.

Now, for each $\alpha \in V^*$, let X_α be the element constructed above. Applying Corollary 10 and Néron-Ogg-Shafarevich, we can find a prime ideal v_α such that $(\rho_V \times \epsilon_\ell)(\pi_{v_\alpha}) = (\alpha, X_\alpha)$, which moreover satisfies

$$\mathrm{Nm}\, v_\alpha \ll_K 4^d \cdot p^6 \cdot (\log N_E + \log p + d)^2 \ll_K 4^d \cdot p^6 \cdot (\log N_E + \log p)^2.$$

(The last inequality follows from Lemma 20, using $a_E \ll_K \log N_E$.) This gives by the Weil bound that for any $\alpha \in V^*$ we can choose v_α so that

$$0 \neq \mathrm{Tr}_E(\pi_{v_\alpha}) \ll_K 2^d \cdot p^3 \cdot (\log N_E + \log p)^2. \tag{4}$$

Now, $\mathrm{Tr}_E(\pi_{v_\alpha})$ must be divisible by all V-exceptional primes ℓ whose corresponding character χ_ℓ satisfies $\chi_\ell(\pi_{v_\alpha}) = -1$. But for any χ_ℓ, half of the $\alpha \in V^*$ satisfy $\chi_\ell(\pi_{v_\alpha}) = -1$. Putting this together,

$$\left(\prod_{\substack{\ell \text{ exceptional} \\ \text{of normalizer type}}} \ell\right)^{2^{d-1}} \;\middle|\; \prod_{\alpha \neq 0 \in V^*} \mathrm{Tr}_E(\pi_{v_\alpha}) \leq \left(c_K \cdot 2^d \cdot p^3 \cdot (\log N_E + \log p + d)\right)^{2^d - 1},$$

where c_K is the effective constant implicit in equation (4). Taking the 2^{d-1}st root of both sides yields the desired conclusion. ∎

5 The Ineffective Bound

Lemma 22. *If p is the smallest acceptable unexceptional prime for an elliptic curve E without CM, then $p \lll_K 1$.*

Proof. By Serre's Open Image Theorem [9], it suffices to verify the statement for all but finitely many elliptic curves E over K. In order to do this, let p be some acceptable prime. In particular, $p \geq 23$, so the genera of the modular curves $X_0(p)$, $X_{\mathrm{split}}(p)$, and $X_{\mathrm{nonsplit}}(p)$ are all at least 2. By Falting's theorem [3], there are finitely many points on each of these modular curves, which completes the proof. ☐

Theorem 23. *Let E be an elliptic curve over a number field K without CM. Then any exceptional prime ℓ satisfies*

$$\ell \lll_K \log N_E.$$

Moreover, the product of all exceptional primes satisfies

$$\prod \ell \lll_K 4^{a_E} \cdot (\log N_E)^{14}.$$

Proof. This is an immediate consequence of Lemmas 17, 21, and 22. ☐

6 The Effective Bound

The bound on the smallest unexceptional prime p in the previous section relies on Falting's theorem, which at the moment is ineffective. Here we give an effective bound on p (which depends on the curve E, but quite gently), using the results of Section 4.

Lemma 24. *Let S be a finite set of primes, p be the smallest acceptable prime number not in S, and b be a constant depending only on K. Then for any A,*

$$\prod_{\ell \in S} \ell \ll_K A \cdot p^b \quad \Rightarrow \quad p \ll_K \log A.$$

Proof. Since the product of all unacceptable primes depends only on K, it suffices to prove this lemma in the case that S contains all of the unacceptable primes. Using (an effective version of) the prime number theorem,

$$p \ll \sum_{\ell < p} \log \ell \leq \log \left(\prod_{\ell \in S} \ell \right) \ll_K \log \left(A \cdot p^b \right) \ll_K \log A + \log p.$$

The desired result follows immediately. ☐

Theorem 25. *Let E be an elliptic curve over a number field K without CM. Then any exceptional prime ℓ satisfies*

$$\ell \ll_K \log N_E \cdot (\log\log N_E)^3.$$

Moreover, the product of all exceptional primes satisfies

$$\prod \ell \ll_K 4^{a_E} \cdot (\log N_E)^{14} \cdot (a_E + \log\log N_E)^6 \cdot (\log\log N_E)^{36}.$$

Proof. From the bound on the product in Lemma 17, together with Lemma 24, we conclude that the smallest prime p not dividing R_E satisfies $p \ll_K \log\log N_E$. Similarly, from the bound on the product in Lemma 21, together with Lemma 24, we conclude that the smallest prime p that is not V-exceptional satisfies $p \ll_K d + \log\log N_E$ (where $d = \dim V$).

Thus, Lemmas 17 and 21 imply the desired result. □

7 Explicit Constants

In this section, we estimate the dependence on K in Theorem 2. Everything used to prove Theorem 2 boils down to the effective Chebotarev theorem (for which the K-dependence is explicit), and Theorem 12. To make Theorem 12 effective, we can use the following result:

Theorem 26. *In Theorem 12, every $\ell \in S_K$ satisfies:*

$$\ell \ll \exp\left(c^{n_K} \cdot (R_K \cdot n_K^{r_K} + h_K \cdot \log \Delta_K)\right);$$

moreover, the product of all $\ell \in S_K$ is bounded by:

$$\prod \ell \ll \exp\left(c^{n_K} \cdot (R_K \cdot n_K^{r_K} + h_K^2 \cdot (\log \Delta_K)^2)\right).$$

Here, c is an effectively computable absolute constant.

Proof. See Theorem 7.9 of [6] for the bound on the product of all $\ell \in S_K$. The bound on the largest element of S_K can be proved in a similar way (just replace $B_{\text{poss}}(K, g, V)$ by 1 in the proof of Theorem 7.9 in [6]). □

Theorem 27. *Let E be an elliptic curve over a number field K without CM. Then any exceptional prime ℓ satisfies*

$$\ell \ll \log N_E \cdot (\log\log N_E)^3 + \exp\left(c^{n_K} \cdot (R_K \cdot n_K^{r_K} + h_K \cdot \log \Delta_K)\right).$$

Moreover, the product of all exceptional primes satisfies

$$\prod \ell \ll 4^{a_E} \cdot (\log N_E)^{13} \cdot (a_E + \log\log N_E)^3 \cdot (\log\log N_E)^{36} \cdot \exp\left(c^{n_K} \cdot (R_K \cdot n_K^{r_K} + h_K^2 \cdot (\log \Delta_K)^2)\right).$$

Here, the constant c and the constants implied by the $\ll$ symbol are all absolute *and effectively computable.*

Proof. This follows from carefully keeping track of the contributions depending on K in the proof of Theorem 2. It is easy to see that the contributions from the bounds given on the set S_K dominate all other contributions coming from the field K. $\square$

References

[1] Yuri Bilu and Pierre Parent. Serre's uniformity problem in the split Cartan case. *Ann. of Math. (2)*, 173(1):569–584, 2011.

[2] Alina Carmen Cojocaru. On the surjectivity of the Galois representations associated to non-CM elliptic curves. *Canad. Math. Bull.*, 48(1):16–31, 2005. With an appendix by Ernst Kani.

[3] G. Faltings. Endlichkeitssätze für abelsche Varietäten über Zahlkörpern. *Invent. Math.*, 73(3):349–366, 1983.

[4] Alain Kraus. Une remarque sur les points de torsion des courbes elliptiques. *C. R. Acad. Sci. Paris Sér. I Math.*, 321(9):1143–1146, 1995.

[5] J. C. Lagarias and A. M. Odlyzko. Effective versions of the Chebotarev density theorem. In *Algebraic number fields: L-functions and Galois properties (Proc. Sympos., Univ. Durham, Durham, 1975)*, pages 409–464. Academic Press, London, 1977.

[6] Eric Larson and Dmitry Vaintrob. Determinants of subquotients of galois representations associated to abelian varieties. Available at `http://arxiv.org/abs/1110.0255`.

[7] H. W. Lenstra, Jr. Algorithms in algebraic number theory. *Bull. Amer. Math. Soc. (N.S.)*, 26(2):211–244, 1992.

[8] D. W. Masser and G. Wüstholz. Galois properties of division fields of elliptic curves. *Bull. London Math. Soc.*, 25(3):247–254, 1993.

[9] Jean-Pierre Serre. Propriétés galoisiennes des points d'ordre fini des courbes elliptiques. *Invent. Math.*, 15(4):259–331, 1972.

[10] Jean-Pierre Serre. Quelques applications du théorème de densité de Chebotarev. *Inst. Hautes Études Sci. Publ. Math.*, (54):323–401, 1981.

[11] Joseph H. Silverman. *The arithmetic of elliptic curves*, volume 106 of *Graduate Texts in Mathematics*. Springer, Dordrecht, second edition, 2009.

[12] David Zywina. Bounds for Serre's open image theorem. Available at `http://arxiv.org/abs/1102.4656`.